UR-1405
ER-40685-852
hep-th/9501110
Jan./1995



# TWO DIMENSIONAL SUPERSYMMETRIC HARMONIC OSCILLATOR CARRYING A REPRESENTATION OF THE GL(2|1) GRADED LIE ALGEBRA.

A.Das[1] and C.Wotzasek[2 3]

*Department of Physics and Astronomy*
*University of Rochester*
*Rochester, NY 14627, USA*

**Abstract**

We study a supersymmetric 2-dimensional harmonic oscillator which carries a representation of the general graded Lie algebra GL(2|1), formulate it on the superspace, and discuss its physical spectrum.

[1]das@urhep.pas.rochester.edu
[2]clovis@urhep.pas.rochester.edu
[3]Permanent address: Instituto de Física, Universidade Federal do Rio de Janeiro, Brasil

# 1 Introduction

Supersymmetry is an interesting symmetry which transforms bosons into fermions and vice-versa [1]. Quantum mechanical (or Classical) theories which are supersymmetric provide realizations of graded Lie algebras (GLA)[2]. The most familiar GLA is the graded Poincaré algebra which leads to relativistic, supersymmetric quantum field theories, which include supergravity.

Examples of simple and extended global supersymmetries based on the grading of space-time symmetries are abundant in simple quantum mechanical systems - the one dimensional supersymmetric harmonic oscillator being the simplest example of such systems[3][4]. There exist, however, many other GLA's which involve grading internal symmetry algebras. The most familiar of such GLA's are the OSp(2m|n) and SL(m|n)[5]. While realizations of such algebras arise naturally in integrable models, there does not yet exist a quantum or classical mechanical realization of the most general graded Lie algebra, namely, GL(m|n). In this paper, we construct a supersymmetric two dimensional harmonic oscillator which provides a realization of GL(2|1) as its symmetry algebra. In section II, we discuss the GL(2|1) algebra with raising and lowering operators. In section III we present our model of a supersymmetric harmonic oscillator and discuss all the symmetries associated with this system. We show that the symmetry algebra coincides with GL(2|1). In section IV, we discuss the spectrum of states associated with this Hamiltonian and present a superspace description of this theory. Finally, we discuss our conclusions in section V.

# 2 Graded Lie Algebra GL(2|1)

Graded Lie algebras[2] include both bosonic and fermionic generators satisfying commutation and anticommutation relations respectively and have the following general structure:

$$\begin{aligned}
[B_m, B_n]_- &= f^k_{mn} B_k \\
[B_m\ , F_\alpha]_- &= h^\beta_{m\alpha} F_\beta \\
[F_\alpha\ \ , F_\beta]_+ &= g^m_{\alpha\beta} B_m
\end{aligned} \tag{1}$$

with the brackets $[\ldots,\ldots]_\mp$ denoting commutators and anti-commutators respectively, $k$, $m$, $n = 1, 2, \ldots, N$, and $\alpha$, $\beta = 1, 2, \ldots, M$. The even or bosonic generators

$B_m$ form the underlying Lie algebra, while the odd or fermionic generators $F_\alpha$ provide a grading of this algebra consistent with the generalized Jacobi identities.

In this section we shall study the graded Lie algebra GL(2|1) whose underlying bosonic algebra is GL(2)$\oplus$GL(1). Here we shall use the boson/fermion representation obtained with help of canonical realizations, i.e., realizations in terms of pairs of boson/fermion creation and annihilation operators satisfying canonical (anti)commutation relations. Consider the set of bosonic and fermionic operators

$$\{a_k^\dagger, a_k \ ; \ k = 1, 2\} \ \text{ and } \ \{a_3^\dagger, a_3\} \tag{2}$$

satisfying the canonical (anti)commutation relations

$$\begin{aligned} \left[a_k, a_m^\dagger\right]_- &= \delta_{km} \\ \left[a_3, a_3^\dagger\right]_+ &= 1 \end{aligned} \tag{3}$$

with all other (anti)commutators vanishing. The four bilinear operators $B \sim a_k^\dagger a_m$ define the generators of the underlying GL(2) algebra,which together with $B \sim a_3^\dagger a_3$ constitute the five bosonic generators of the GL(2|1) algebra. The four fermionic generators of this algebra are defined by the bilinear operators $F \sim a_3^\dagger a_k$ and $F \sim a_k^\dagger a_3$. All together these nine operators generate the $Z_2$ graded GL(2|1) algebra. It is a simple task to verify that they satisfy the algebra (1). For instance, if we denote these nine operators as

(a) Bosonic Generators

$$\begin{aligned} B_1 &= a_1^\dagger a_1 \\ B_2 &= a_1^\dagger a_2 \\ B_3 &= a_2^\dagger a_1 \\ B_4 &= a_2^\dagger a_2 \\ B_5 &= a_3^\dagger a_3 \end{aligned} \tag{4}$$

(b) Fermionic Generators

$$\begin{aligned} F_1 &= a_1^\dagger a_3 \\ F_2 &= a_2^\dagger a_3 \end{aligned}$$

$$
\begin{aligned}
F_3 &= a_3^\dagger a_1 \\
F_4 &= a_3^\dagger a_2
\end{aligned} \tag{5}
$$

it is then a simple task to find the nonvanshing structure constants in (1).

For future convenience, we introduce a new basis of the fermionic generators as

$$
\begin{aligned}
Q_{R} &= \frac{1}{\sqrt{2}}\left(a_1^\dagger a_3 - i a_2^\dagger a_3\right) \\
\overline{Q}_{R} &= \frac{1}{\sqrt{2}}\left(a_3^\dagger a_1 + i a_3^\dagger a_2\right) \\
Q_{L} &= \frac{i}{\sqrt{2}}\left(a_3^\dagger a_1 - i a_3^\dagger a_2\right) \\
\overline{Q}_{L} &= \frac{-i}{\sqrt{2}}\left(a_1^\dagger a_3 + i a_2^\dagger a_3\right)
\end{aligned} \tag{6}
$$

The anti-commutation relations amongst these charges are easily computed. We also redefine the five bosonic operators as

$$
\begin{aligned}
h_b &= a_1^\dagger a_1 + a_2^\dagger a_2 \\
h_f &= a_3^\dagger a_3 \\
\Delta_1 &= a_1^\dagger a_1 - a_2^\dagger a_2 \\
i\,\Delta_2 &= a_2^\dagger a_1 - a_1^\dagger a_2 \\
\Delta_3 &= a_1^\dagger a_2 + a_2^\dagger a_1
\end{aligned} \tag{7}
$$

and introduce the operator $H = h_b + h_f$. The algebra of the fermionic charges (6) becomes

$$
\begin{aligned}
\left[Q_{R}, \overline{Q}_{R}\right]_+ &= \frac{1}{2}\left(H + \Delta_2 + h_f\right) \\
\left[Q_{L}, \overline{Q}_{L}\right]_+ &= \frac{1}{2}\left(H - \Delta_2 + h_f\right) \\
\left[Q_{R}, Q_{L}\right]_+ &= \frac{1}{2}\left(\Delta_3 + i\,\Delta_1\right) \\
\left[\overline{Q}_{R}, \overline{Q}_{L}\right]_+ &= \frac{1}{2}\left(\Delta_3 - i\,\Delta_1\right)
\end{aligned} \tag{8}
$$

Similarly, the algebra of the new bosonic charges (7) can be computed straightforwardly, to give

$$\begin{aligned}
[\Delta_k, \Delta_m]_- &= 2\, i\, \epsilon_{kmn}\, \Delta_n \\
[h_b, \Delta_m]_- &= 0 \\
[h_f, \Delta_m]_- &= 0 \\
[h_b, h_f]_- &= 0
\end{aligned} \tag{9}$$

while the remaining nonvanishing boson-fermion commutation relations are

$$\begin{aligned}
[h_f, Q_R]_- &= -\, [h_b, Q_R]_- = -\, Q_R \\
\left[h_f, \overline{Q}_R\right]_- &= -\, \left[h_b, \overline{Q}_R\right]_- = +\, \overline{Q}_R \\
[h_f, Q_L]_- &= -\, [h_b, Q_L]_- = +\, Q_L \\
\left[h_f, \overline{Q}_L\right]_- &= -\, \left[h_b, \overline{Q}_L\right]_- = -\, \overline{Q}_L
\end{aligned} \tag{10}$$

$$\begin{aligned}
[\Delta_1, Q_R]_- &= +i\, \overline{Q}_L \\
\left[\Delta_1, \overline{Q}_R\right]_- &= +i\, Q_L \\
[\Delta_1, Q_L]_- &= -i\, \overline{Q}_R \\
\left[\Delta_1, \overline{Q}_L\right]_- &= -i\, Q_R
\end{aligned} \tag{11}$$

$$\begin{aligned}
[\Delta_2, Q_R]_- &= +\, Q_R \\
\left[\Delta_2, \overline{Q}_R\right]_- &= -\, \overline{Q}_R \\
[\Delta_2, Q_L]_- &= +\, Q_L \\
\left[\Delta_2, \overline{Q}_L\right]_- &= -\, \overline{Q}_L
\end{aligned} \tag{12}$$

$$\begin{aligned}
[\Delta_3, Q_R]_- &= +\, \overline{Q}_L \\
\left[\Delta_3, \overline{Q}_R\right]_- &= -\, Q_L \\
[\Delta_3, Q_L]_- &= -\, \overline{Q}_R \\
\left[\Delta_3, \overline{Q}_L\right]_- &= +\, Q_R
\end{aligned} \tag{13}$$

Using these relations we can verify that all generators satisfy the generalized Jacobi identities. Observe from eqs.(9) and (10) that all nine generators of the GL(2|1) graded Lie algebra described above commute with $H$ which stays in the center of the algebra.

# 3 Two Dimensional Supersymmetric Harmonic Oscillator

In this section we introduce our model, a two-dimensional supersymmetric harmonic oscillator which, as mentioned in the introduction, carries a representation of the graded GL(2|1) algebra described in section II. This model is described by the following Lagrangian

$$L = \frac{1}{2}\left(\dot{q}^T \dot{q} - q^T q\right) + \frac{i}{2}\psi^T \left(\frac{d}{dt} - i\,\sigma_2\right)\psi \tag{14}$$

where

$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \tag{15}$$

and

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{16}$$

are the oscillator's bosonic and fermionic coordinates, which are assumed to be real. We have chosen the mass and the frequency to be unity, for simplicity. Here $\sigma_k$ stands for the Pauli matrices and $q^T$ and $\psi^T$ stand for matrix transposition, as usual. Note that up to total derivatives, we can also write the Lagrangian (14) as

$$L = -\frac{1}{2}q^T \left(\frac{d}{dt} - i\,\sigma_2\right)\left(\frac{d}{dt} + i\,\sigma_2\right) q + \frac{i}{2}\psi^T \left(\frac{d}{dt} - i\,\sigma_2\right)\psi \tag{17}$$

It is important to mention that compared to the *usual* two dimensional supersymmetric harmonic oscillator, the model under investigation here is constructed with half the number of fermionic degrees of freedom, i.e, it has two second-order bosonic variables (or four first-order) and two first-order fermionic variables. In the two-dimensional matrix space of Eqs.(15,16), let us denote a complete basis of real, (2x2) matrices by

$$\tau_a = (\sigma_0, \sigma_1, i\,\sigma_2, \sigma_3)\ ;\ \ a = 0,1,2,3 \tag{18}$$

where $\sigma_0$ is the (2x2) identity matrix. It is straightforward to show that the action of the theory described by (14) is invariant under the four supersymmetry transformations

$$\begin{aligned} \delta q &= \varepsilon_a \tau_a \psi \\ \delta\psi &= i\left(\frac{d}{dt} + i\,\sigma_2\right)\varepsilon_a \tau_a^T q \end{aligned} \tag{19}$$

with $\varepsilon_a$ being four infinitesimal, constant Grassmann parameters that characterize the transformations. In the Hamiltonian language[6], which is more appropriate for our purposes, the Hamiltonian operator is given by

$$H = \frac{1}{2}\left(p^T p + q^T q\right) - \frac{1}{2}\psi^T \sigma_2 \psi \tag{20}$$

and enjoys the following set of global invariances
(A) Supersymmetry:

$$\begin{aligned} \delta q &= \frac{1}{\sqrt{2}}\varepsilon_a \tau_a \psi \\ \delta\psi &= \frac{i}{\sqrt{2}}\varepsilon_a\left(\tau_a^T p + i\,\sigma_2 \tau_a^T q\right) \\ \delta p &= \frac{i}{\sqrt{2}}\varepsilon_a \tau_a \sigma_2 \psi \end{aligned} \tag{21}$$

The Noether supercharges generating these transformations are given by

$$Q_a = \frac{1}{\sqrt{2}}\left(p^T \tau_a \psi - i\,q^T \tau_a \sigma_2 \psi\right) \tag{22}$$

Indeed, note that given the generalized Dirac brackets (see Eq.(A.5))

$$\begin{aligned} \{q_k, p_m\} &= \delta_{km} \\ \{\psi_\alpha, \psi_\beta\} &= -i\,\delta_{\alpha\beta} \end{aligned} \tag{23}$$

we obtain the supersymmetry transformations (21) above as

$$\delta A = \{A, \varepsilon_a Q_a\} \tag{24}$$

where $A$ stands for $q_k$, $p_k$ and $\psi_\alpha$. The invariance of the Hamiltonian implies that $\{H, Q_a\} = 0$ which in turn shows that the $Q_a$'s are constants of motion. Besides the four supersymmetries above, this Hamiltonian is also invariant under the following global symmetries.

(B) Rotation on the four-dimensional (bosonic) phase-space plane:

(i) The transformations

$$\begin{aligned} \delta q &= -i\ \alpha \sigma_2 q \\ \delta p &= -i\ \beta \sigma_2 p \end{aligned} \tag{25}$$

with $\alpha$,$\beta$ bosonic, constant infinitesimal parameters clearly are a symmetry of $H$. However, in order to preserve the canonical commutation relations (23) we must have $\alpha = \beta$. These transformations are generated by the angular momentum operator

$$J_1 = \frac{i}{2}\left(q^T \sigma_2 p - p^T \sigma_2 q\right) \tag{26}$$

Note that these transformations do not mix coordinate and momentum variables.

(ii) The transformations

$$\begin{aligned} \delta q &= \lambda_a \tau_a p \\ \delta p &= -\lambda_a \tau_a q \end{aligned} \tag{27}$$

with $\lambda_a$ constant, bosonic, infinitesimal parameters are also a set of symmetries of the Hamiltonian which preserve the Dirac brackets relations, as long as $a \neq 2$. The charges generating these transformations are

$$\begin{aligned} L_0 &= \frac{1}{2}\left(p^T p + q^T q\right) \\ L_1 &= \frac{1}{2}\left(p^T \sigma_1 p + q^T \sigma_1 q\right) \\ L_3 &= \frac{1}{2}\left(p^T \sigma_3 p + q^T \sigma_3 q\right) \end{aligned} \tag{28}$$

(C) Rotation on the two-dimensional fermionic phase-space.

The generalized rotation in the fermionic phase-space

$$\delta\psi = \vec{\alpha}\cdot\vec{\sigma}\ \psi \tag{29}$$

with $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ being a set of constant bosonic parameters, is another group of symmetries in the action. Again, preservation of the canonical commutation relations (23) imposes $\alpha_1 = \alpha_3 = 0$ as conditions over the possible values that these parameters can take. These transformations are generated by the following charge,

$$J_2 = \frac{1}{2}\psi^T\sigma_2\psi \tag{30}$$

representing the fermionic contribution to the total angular momentum.

Now, in order to make contact with the graded algebra GL(2|1) defined in the preceeding section, let us redefine these charges as

$$\begin{aligned}
h_b &= L_0 = \frac{1}{2}\left(p^Tp + q^Tq\right)\\
h_f &= -J_2 = -\frac{1}{2}\psi^T\sigma_2\psi\\
\Delta_1 &= L_1 = \frac{1}{2}\left(p^T\sigma_1 p + q^T\sigma_1 q\right)\\
\Delta_2 &= J_1 = \frac{i}{2}\left(q^T\sigma_2 p - p^T\sigma_2 q\right)\\
\Delta_3 &= L_3 = \frac{1}{2}\left(p^T\sigma_3 p + q^T\sigma_3 q\right)
\end{aligned} \tag{31}$$

The canonical Dirac bracket algebra of these charges read

$$\begin{aligned}
\{\Delta_k, \Delta_m\} &= 2\ \epsilon_{kmn}\Delta_n\\
\{h_b, \Delta_k\} &= 0\\
\{h_f, \Delta_k\} &= 0
\end{aligned} \tag{32}$$

and can be seen to agree with the bosonic sector of the GL(2|1) algebra, eq.(9), when the classical Dirac bracket algebra is quantized through the usual replacement $\{\ldots,\ldots\} \rightarrow -i\ [\ldots,\ldots]_{\mp}$.

Next we redefine the supersymmetry generators (22) as

$$
\begin{aligned}
Q_{R} &= \frac{1}{2}\left(Q_0 + i\, Q_2\right) \\
&= \frac{1}{\sqrt{2}}\left(p^T + i\, q^T\right) T_+ \psi \\
\overline{Q}_{R} &= \frac{1}{2}\left(Q_0 - i\, Q_2\right) \\
&= \frac{1}{\sqrt{2}}\left(p^T - i\, q^T\right) T_- \psi \\
Q_{L} &= \frac{1}{2}\left(Q_1 + i\, Q_3\right) \\
&= \frac{1}{\sqrt{2}}\left(p^T - i\, q^T\right) \sigma_1 T_+ \psi \\
\overline{Q}_{L} &= \frac{1}{2}\left(Q_1 - i\, Q_3\right) \\
&= \frac{1}{\sqrt{2}}\left(p^T + i\, q^T\right) \sigma_1 T_- \psi
\end{aligned}
\tag{33}
$$

where $T_{\pm} = \frac{1}{2}\left(1 \mp \sigma_2\right)$ is the "helicity" projection operator. Using the genralized Dirac brackets, Eq.(23), we can verify that the nine symmetry generating operators in (31) and (33), possess an algebra whose quantum version is isomorphic to that presented in section II. Moreover, if we introduce, as usual, the representation of the phase-space variables $q_k$ and $p_k$ in terms of creation and annihilation operators as

$$
\begin{aligned}
a_k &= \frac{1}{\sqrt{2}}\left(q_k + i\, p_k\right) \\
a_k^{\dagger} &= \frac{1}{\sqrt{2}}\left(q_k - i\, p_k\right)
\end{aligned}
\tag{34}
$$

and define

$$
\begin{aligned}
a_3 &= \frac{i}{\sqrt{2}}\left(\psi_1 + i\, \psi_2\right) \\
a_3^{\dagger} &= \frac{-i}{\sqrt{2}}\left(\psi_1 - i\, \psi_2\right)
\end{aligned}
\tag{35}
$$

which, by virtue of (23) do satisfy (3), then we can write the generators of symmetry in (31) and (33) in the same form as the graded Lie algebra generators (6) and (7), defined in section II. It becomes a matter of simple calculation to verify that these nine charges (five bosonic and four fermionic) satisfy exactly the graded Lie Algebra GL(2|1) found in section II.

# 4 Spectrum And Superspace Formulation

Let us examine in this section the action of the GL(2|1) operators, defined in the previous sections, on the states of the Hilbert space of the quantum mechanical model. The spectrum of the normal ordered theory is given by $\{\mathcal{E}_n, |\, n >\}$, where the eigenvalues and eigenvectors are

$$\begin{aligned} \mathcal{E}_n &= n_+ + n_- + n_f \\ |\, n > &= |\, n_+, n_-, n_f > \end{aligned} \tag{36}$$

with $n_\pm = 1, 2, \ldots$ and $n_f = 0, 1$. Conventionally, the states with $n_f = 0(1)$ are called bosonic (fermionic). Here $n_\pm$ and $n_f$ are the eigenvalues of the bosonic and fermionic number operators, $N_\pm = a^\dagger_\pm a_\pm$ and $N_f = a^\dagger_3 a_3$, and

$$a_\pm = \frac{1}{\sqrt{2}}\,(a_1 \pm i\, a_2) \tag{37}$$

By inspection we see that the ground state is a non-degenerate bosonic state with zero energy. The first excited level is three fold degenerate, possessing one fermionic and two bosonic states. The second excited energy level is five fold degenerate, with two fermionic and three bosonic states, and so on. The states of the first few levels are displayed below:

$$\begin{aligned} |\, 0 > &= |\, 0,0,0 > \\ |\, 1 > &= \{|\, 1,0,0 >,\, |\, 0,0,1 >,\, |\, 0,1,0 >\} \\ |\, 2 > &= \{|\, 2,0,0 >,\, |\, 1,0,1 >,\, |\, 1,1,0 >,\, |\, 0,1,1 >,\, |\, 0,2,0 >\} \\ &\;\vdots \end{aligned} \tag{38}$$

In terms of the (chiral) operators (37), the GL(2|1) generators (6) and (7) read

$$
\begin{aligned}
Q_{R} &= a_+^\dagger a_3 \\
\overline{Q}_{R} &= a_+ a_3^\dagger \\
Q_{L} &= i\, a_- a_3^\dagger \\
\overline{Q}_{L} &= -i\, a_-^\dagger a_3 \\
\Delta_1 &= a_+^\dagger a_- + a_-^\dagger a_+ \\
\Delta_2 &= a_+^\dagger a_+ - a_-^\dagger a_- \\
i\,\Delta_3 &= a_-^\dagger a_+ - a_+^\dagger a_- \\
h_b &= a_+^\dagger a_+ + a_-^\dagger a_- \\
h_f &= a_3^\dagger a_3
\end{aligned}
\tag{39}
$$

On the degenerate levels the supersymmetry generators take bosonic states into fermionic ones and vice versa as

$$
\begin{aligned}
Q_{R} \mid n_+, n_-, n_f > &= \sqrt{n_+ + 1}\; \delta_{n_f,1} \mid n_+ + 1, n_-, n_f - 1 > \\
\overline{Q}_{R} \mid n_+, n_-, n_f > &= \sqrt{n_+}\; \delta_{n_f,0} \mid n_+ - 1, n_-, n_f + 1 > \\
Q_{L} \mid n_+, n_-, n_f > &= \sqrt{n_-}\; \delta_{n_f,0} \mid n_+, n_- - 1, n_f + 1 > \\
\overline{Q}_{L} \mid n_+, n_-, n_f > &= \sqrt{n_- + 1}\; \delta_{n_f,1} \mid n_+, n_- + 1, n_f - 1 >
\end{aligned}
\tag{40}
$$

Quite clearly the supersymmetry charge $Q_{R}$ ($\overline{Q}_{R}$) creates (destroys) a right-handed boson and destroys (creates) a fermion. Similarly the $Q_{L}$ ($\overline{Q}_{L}$) destroys (creates) a left handed boson while creating (destroying) a fermion. Note that $Q_{R}$ and $\overline{Q}_{R}$ ($Q_{L}$ and $\overline{Q}_{L}$) have no effect on the $n_-$ ($n_+$) eigenvalues, showing the existence of a chiral supersymmetry which, ultimately, is due to the fact that we have twice as many bosonic variables as the fermionic ones. The supersymmetry charges can be used to generate the states in a given level once the highest state is given. Then, starting from the state $\mid 0, n, 0 >$, one can generate all the states belonging to the $\mathcal{E}_n$ subspace by consecutive applications of $Q_{L}$ and $Q_{L}$ until the state $\mid n, 0, 0 >$ is reached.

$$
\mid n,0,0 > \xleftarrow{Q_R} \mid n-1,0,1 > \xleftarrow{Q_L} \mid n-1,1,0 > \ldots \mid 0, n-1, 1 > \xleftarrow{Q_L} \mid 0,n,0 > \tag{41}
$$

Similarly, starting with the state $\mid n, 0, 0 >$ and using consecutively the charges $\overline{Q}_{R}$ and $\overline{Q}_{L}$ one generates the whole subspace $\mathcal{E}_n$ again.

$$\mid n,0,0 > \xrightarrow{\overline{Q}_R} \mid n-1,0,1 > \xrightarrow{\overline{Q}_L} \mid n-1,1,0 > \ldots \mid 0,n-1,1 > \xrightarrow{\overline{Q}_L} \mid 0,n,0 > \quad (42)$$

This action of the supersymmetry charges is easily seem on the set of states shown in equation (38).

The action of the bosonic operators on the other hand only connect states with the same fermion number. The operators $h_f$, $h_b$, and $\Delta_2$ are diagonal in the chiral basis (36):

$$\begin{aligned} h_b \mid n_+, n_-, n_f > &= (n_+ + n_-) \mid n_+, n_-, n_f > \\ \Delta_2 \mid n_+, n_-, n_f > &= (n_+ - n_-) \mid n_+, n_-, n_f > \\ h_f \mid n_+, n_-, n_f > &= n_f \mid n_+, n_-, n_f > \end{aligned} \quad (43)$$

These operators have the usual interpretation as bosonic and fermionic Hamiltonians ($h_b$ and $h_f$), and chirality operator ($\Delta_2$). Finally, the non-diagonal operators $\Delta_1$ and $\Delta_3$ being bosonic in nature only connect states of the same Grassman parity but with chiralities two unities different from the initial state:

$$\begin{aligned} \frac{1}{2}(\Delta_1 + i\Delta_3) \mid n_+, n_-, n_f > &= \sqrt{(n_- + 1)n_+} \mid n_+ - 1, n_- + 1, n_f > \\ \frac{1}{2}(\Delta_1 - i\Delta_3) \mid n_+, n_-, n_f > &= \sqrt{(n_+ + 1)n_-} \mid n_+ + 1, n_- - 1, n_f > \end{aligned} \quad (44)$$

We finish this section with a discussion of the superspace formulation of this problem. To this end we rewrite the supersymmetry transformations (19) in terms of the transformations generated by the chiral supersymmetry charges (33), which seem to be more appropriate for this model. The transformations generated by $Q_R$, $\overline{Q}_R$, $Q_L$ and $\overline{Q}_L$ are, respectively

$$\begin{aligned} \delta_R\, q &= \varepsilon_R\, T_+\, \psi \\ \delta_R\, \psi &= \varepsilon_R\, T_-\, \mathcal{D}_+ q \end{aligned} \quad (45)$$

$$\begin{aligned} \overline{\delta}_R\, q &= \overline{\varepsilon}_R\, T_-\, \psi \\ \overline{\delta}_R\, \psi &= \overline{\varepsilon}_R\, T_+\, \mathcal{D}_- q \end{aligned} \quad (46)$$

$$
\begin{aligned}
\delta_L \, q &= \varepsilon_L \, T_+ \, \sigma_1 \psi \\
\delta_L \, \psi &= \varepsilon_L \, T_+ \, \sigma_1 \mathcal{D}_- q
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
\overline{\delta}_L \, q &= \overline{\varepsilon}_L \, T_- \, \sigma_1 \psi \\
\overline{\delta}_R \, \psi &= \overline{\varepsilon}_L \, T_- \, \sigma_1 \mathcal{D}_+ q
\end{aligned} \tag{48}
$$

Here we have introduced the notation $\mathcal{D}_\pm = i(\partial_t \pm i)$. To obtain these four supersymmetries in a superfield language, we introduce two Grassman variables for each chiral sector as $\theta_R$, $\overline{\theta}_R$, $\theta_L$ and $\overline{\theta}_L$, and define two chiral superfields $\phi_R$ and $\phi_L$. The transformation in the right chiral sector can be obtained from the following superfield and (differential operator) supercharge

$$
\begin{aligned}
\phi_R &= q + \theta_R T_+ \psi + \overline{\theta}_R T_- \psi \\
Q_R &= T_+ \frac{\partial}{\partial \theta_R} - \overline{\theta}_R T_- \mathcal{D}_+ \\
\overline{Q}_R &= T_- \frac{\partial}{\partial \overline{\theta}_R} - \theta_R T_+ \mathcal{D}_-
\end{aligned} \tag{49}
$$

while those in the left chiral sector come from

$$
\begin{aligned}
\phi_L &= q + \theta_L T_+ \sigma_1 \psi + \overline{\theta}_L T_- \sigma_1 \psi \\
Q_L &= T_+ \frac{\partial}{\partial \theta_L} - \overline{\theta}_L T_- \mathcal{D}_- \\
\overline{Q}_L &= T_- \frac{\partial}{\partial \overline{\theta}_L} - \theta_L T_+ \mathcal{D}_+
\end{aligned} \tag{50}
$$

These transformations can be organized in a matrix like structure with the following form

$$
\begin{aligned}
\delta \Phi &= \epsilon^T \mathbf{Q} \Phi \\
\overline{\delta} \Phi &= \overline{\epsilon}^T \overline{\mathbf{Q}} \Phi
\end{aligned} \tag{51}
$$

where

$$\mathbf{Q} = \begin{pmatrix} Q_R & 0 \\ 0 & Q_L \end{pmatrix} \tag{52}$$

$$\overline{\mathbf{Q}} = \begin{pmatrix} \overline{Q}_R & 0 \\ 0 & \overline{Q}_L \end{pmatrix} \tag{53}$$

are block-diagonal (4x4) matrices and

$$\Phi = \begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix} \tag{54}$$

$$\epsilon = \begin{pmatrix} \varepsilon_R \\ \varepsilon_L \end{pmatrix} \tag{55}$$

are (4x1) column matrices. We notice here that a matrix structure is essential for the superspace formulation since the GLA, in this case, grades an internal symmetry algebra (for example, Eq.(8) involves not just the Hamiltonian, but the internal symmetry generators as well which would have a matrix representation). The matrix structure of $Q$ would depend on the internal space upon which it acts (unlike the usual space-time supersymmetry charges) and the form given here is appropriate only for the doublet space of $q$ and $\psi$. Next we introduce the four covariant derivatives as

$$\begin{aligned} D_R &= T_- \frac{\partial}{\partial \theta_R} + \overline{\theta}_R T_+ \mathcal{D}_+ \\ \overline{D}_R &= T_+ \frac{\partial}{\partial \overline{\theta}_R} - \theta_R T_- \mathcal{D}_- \end{aligned} \tag{56}$$

$$\begin{aligned} D_L &= T_- \frac{\partial}{\partial \theta_L} + \overline{\theta}_L T_+ \mathcal{D}_- \\ \overline{D}_L &= T_+ \frac{\partial}{\partial \overline{\theta}_L} - \theta_L T_- \mathcal{D}_+ \end{aligned} \tag{57}$$

and define

$$\mathbf{D} = \begin{pmatrix} D_R & 0 \\ 0 & D_L \end{pmatrix} \tag{58}$$

$$\overline{\mathbf{D}} = \begin{pmatrix} \overline{D}_R & 0 \\ 0 & \overline{D}_L \end{pmatrix} \tag{59}$$

These covariant derivatives can be easily seen to anticommute with all supersymmetry charges $\mathbf{Q}$ and $\overline{\mathbf{Q}}$. In terms of these covariant derivatives, the Lagrangian of this theory can be written as

$$\begin{aligned} L &= \frac{1}{2}\sum_{A=R,L}\int d\theta_A d\overline{\theta}_A \left[\left(\overline{\mathbf{D}}\Phi\right)^T \cdot (\mathbf{D}\Phi) - \Phi^T \cdot \overline{\mathbf{D}}\mathbf{D}\Phi\right] \\ &= \frac{1}{2}\sum_{A=R,L}\int d\theta_A d\overline{\theta}_A \left[\left(\overline{D}_A\phi_A\right)^T \cdot (D_A\phi_A) - \phi_A^T \cdot \overline{D}_A D_A\phi_A\right] \end{aligned} \tag{60}$$

# 5 Conclusion

In this work we have studied a supersymmetric harmonic oscillator possessing twice as many bosonic variables than fermionic ones. The model enjoys a chiral supersymmetry when the fermionic variables are interchanged with either one of the chiral bosonic sectors. Besides the supersymmetries, we have worked out all the global symmetries of the model and verified that the generators provide a representation of the general graded Lie algebra GL(2|1). We have worked out the physical spectrum of this model and constructed the superspace formulation. It is interesting to see that in the superspace language the separation of the chiral sectors are clearly displayed, and the charges and the covariant derivatives carry a matrix structure essentially because the algebra represents the grading of an internal symmetry group.
**ACKNOWLEDGEMENTS** This work has been supported in part by U.S. Department of Energy, grant No DE-FG-02-91ER 40685, and by CNPq, Brazilian research agency, Brasilia, Brazil.

# A Appendix: Dirac Brackets Via Faddeev-Jackiw Formalism

First order Lagrangians are constrained systems and must be quantized with Dirac brackets instead of Poisson brackets[7][8]. The Dirac brackets of an arbitray first-order system can be calculated with easy using the technique put forward by Faddeev and

Jackiw a few years ago[9]. Consider an arbitray system with a finite number of degrees of freedom $Z_A$, whose Grassman parity is $\epsilon_A$ and is described by

$$L = \dot{Z}_A K_A(Z_A) - V(Z_A) \tag{A.1}$$

The equations of motion read

$$\dot{Z}_B M_{BA} = -\frac{\partial V}{\partial Z_A} \tag{A.2}$$

where

$$M_{AB} = \frac{\partial K^B}{\partial Z_A} - (-1)^{\epsilon_A \epsilon_B} \frac{\partial K^A}{\partial Z_B} \tag{A.3}$$

is the generalized sympletic matrix[10]. If the sympletic matrix is nonsingular, the equation of motion (A.2) can be solved for the velocities as

$$\dot{Z}_A = (-1)^{\epsilon_A} M^{-1}_{AB} \frac{\partial V}{\partial Z_B} \tag{A.4}$$

and be written in Hamiltonian form with the introduction of some generalized or Dirac bracket as

$$\{Z_A\ ,\ Z_B\} = (-1)^{\epsilon_A} M^{-1}_{AB} \tag{A.5}$$

The equations of motion then take the following form

$$\dot{Z}_A = \{Z_A\ ,\ V(Z)\} \tag{A.6}$$

Using (A.5) one can verify that the Dirac brackets for the fermionic variables of the supersymmetric two-dimensional oscillator are those given in eq.(23)[11].

# References


[1] For reviews in supersymmetry and supergravity, as well as references on the original papers see, for instance, P.Fayet and S.Ferrara, Phys.Rep.**32C**(1977)249; J.Wess and J.Bagger, "Supersymmetry and Supergravity", Princeton University Press, Princeton, NJ 1983; S.J.Gates, M.T.Grisaru, M.Rocek and W.Siegel, "Superspace, or One thousend and One Lessons in Supersymmetry", Benjamin/Cunnings, Reading, Mass. 1983,; P.van Nieuwenhuisen, Phys.Rep.**68C**(1981)264; M.F.Sohnius, Phys.Rep.**128C**(1985)40.

[2] P.G.O.Freund and I.Kaplansky, Jour.Math.Phys. **17**(1976)228.

[3] E.Witten, Nucl.Phys.**B188**(1981)513.

[4] P.Salomonson and J.W.van Holten, Nucl.Phys.**B196**(1982)509 and M.de Combrugghe and V.Rittenberg, Annals of Physics, **151**(1983)99.

[5] A.Das and S.Roy, Jour.Math.Phys.**31**(1990)2145, and A.Das, W.J.Huang and S.Roy, Intl.Jour.Mod.Phys.**A7**(1992)4293.

[6] J.Barcelos-Neto and A.Das, Phys.Rev.**D33**(1986)2863.

[7] P.A.M.Dirac, Can.J.Math.**2**(1950)129; Lectures on Quantum Mechanics (Belfer Graduate School of Science, Yeshiva University, New York, 1964).

[8] For a general review, see A. Hanson, T. Regge, and C. Teitelboim, Constrained Hamiltonian Systems (Academia Nazionale dei Lincei, Rome, 1976).

[9] L.D.Faddeev and R.Jackiw, Phys.Rev.Lett.**60**(1988)587.

[10] J.Govaerts, Intl.J.Mod.Phys.**A5**(1990)3625.

[11] J.Barcelos-Neto and E.S.Cheb-Terrab, Z.Phys.**C54**(1992) 133.